NONSTIFFNESS OF SPHERICAL SHELLS

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The paper [1] introduced a class of nonstiff shells, i.e. shells which for certain types of support have nontrivial equilibrium configurations in the absence of exterior loads. As the definition implies, the characteristic property of the nonstiff shells consists of the fact that the lowest critical load for such shells is a negative quantity. Herein, we obtain rigorous proof of the existence of nonstiff shells. Namely, it is shown that, for a thin spherical shell with immovable, hinged support at the boundary, there exists another equilibrium configuration close to a mirror image. The proof employs the asymptotic method developed in [2 and 3].

1. Formulation of the Problem. Consider the system of nonlinear differential Eqs. of an unloaded spherical shell [4 and 5]

$$\varepsilon^2 Av - {}^1/_2 u^2 + \rho u = 0, \qquad \varepsilon^2 Au + uv - \rho v = 0$$
$$A(\cdots) \equiv -\rho \frac{d}{d\rho}\frac{1}{\rho}\frac{d}{d\rho}\rho(\ ), \qquad 0 \leqslant \rho \leqslant 1, \qquad 0 < \mu < \frac{1}{2} \tag{1.1}$$

with the boundary conditions

$$\left[\frac{dv}{d\rho} - \frac{\mu}{\rho} v\right]_{\rho=1} = 0, \quad \left[\frac{du}{d\rho} + \frac{\mu}{\rho} u\right]_{\rho=1} = 0; \qquad \frac{v}{\rho}\bigg|_{\rho=0} < \infty, \qquad \frac{u}{\rho}\bigg|_{\rho=0} < \infty \tag{1.2}$$

All quantities in (1.1) and (1.2) have been nondimensionalized, with

$$u = \frac{R}{a}\frac{dw}{dr}, \quad v = \frac{\gamma R}{ahE}\frac{dF}{dr}, \quad \varepsilon^2 = \frac{hR}{a^2\gamma}, \quad \gamma^2 = 12(1-\mu^2)$$

Here, w is the deflection of the shell middle surface, F is a stress function, E is Young's modulus, μ is Poisson's ratio, h is the shell thickness, a is the radius of the exterior surface, R is the shell radius and $r = a\rho$. The small parameter ε^2 characterizes the shell wall thickness. The boundary conditions correspond to a condition of hinged, immovable support.

It is easily seen that the problem posed by (1.1), (1.2) has the trivial solution $v = u \equiv 0$. This solution corresponds to an equilibrium form with zero stresses and strains. The question arises whether or not this form is unique; a study of very thin shells shows that it is not. For example, the hollow shape of a poorly inflated ball is retained after the pressure causing it has been removed. We will attempt to explain this fact with the aid of (1.1). Since we are concerned with very thin shells, we will consider small values of ε^2.

Setting $\varepsilon = 0$, we obtain the algebraic Eqs.

$$-\frac{u_0^2}{2} + \rho u_0 = 0, \qquad u_0 v_0 - \rho v_0 = 0 \tag{1.3}$$

There are two solutions. One of these is $u_0 = v_0 \equiv 0$, the trivial solution which also satisfies (1.1) and (1.2). The second solution

$$v_0 = 0, \qquad u_0 = 2\rho \tag{1.4}$$

corresponds to an equilibrium form which is close to a mirror image.

The solution (1.4) satisfies (1.1), but does not satisfy the second boundary condition in (1.2). Thus, one would expect that, for small ε, the problem (1.1), (1.2) has a second solution which behaves like (1.4) everywhere inside the region, but when it approaches the boundary it undergoes a rapid change so as to satisfy the boundary conditions (1.2).

In order to show the existence of a second solution, we will first construct the asymptotic expansions for small ε in the neighborhood of (1.4) (Section 2), and then we will show the existence of a solution to (1.1), (1.2) for which these asymptotic expansions hold (Section 3). Here we make use of a theorem from [2 and 3] which has previously been employed in connection with asymptotic solutions of some nonlinear problems. Finally, in Section 4, we study an example and present curves for the fundamental characteristic of the second form of equilibrium.

Note that the existence of nonstiff shells under the boundary conditions (1.2) was also confirmed by a detailed numerical analysis of this problem in [6].

The asymptotic analysis of the problem given below clarifies to some extent the essence of certain hypotheses of Pogorelov [7].

2. Construction of the asymptotic expansions. Introduce the following notation: Let the vector $V \equiv (v, u)$ be the solution and let $P[V]$ be the left-hand side of (1.1). For the second solution, we construct the asymptotic expansions (2.1)

$$v = \sum_{s=0}^{n} \varepsilon^s v_s + \sum_{s=0}^{n} \varepsilon^s h_s + \sum_{s=0}^{n} \varepsilon^s \alpha_s + x_n, \qquad u = \sum_{s=0}^{n} \varepsilon^s u_s + \sum_{s=0}^{n} \varepsilon^s g_s + \sum_{s=0}^{n} \varepsilon^s \beta_s + z_n$$

The functions $v_s(\rho)$ and $u_s(\rho)$ are obtained with the aid of the first iterative procedure [8]. Namely, we require that

$$\mathbf{P}[\mathbf{V}_n] = O(\varepsilon^{n+1}), \qquad \mathbf{V}_n \equiv \Big(\sum_{s=0}^{n} \varepsilon^s v_s, \ \sum_{s=0}^{n} \varepsilon^s u_s\Big)$$

We set the coefficients of the various powers of ε equal to zero, and we obtain (1.3) for the determination of v_0 and u_0 (for which we choose the second solution, (1.4)); for the determination of v_s and u_s, we have a system of homogeneous linear equations. Thus,

$$v_s(\rho) = u_s(\rho) = 0 \qquad (s = 1, 2, \ldots, n)$$

Boundary layer type functions $h_s(\rho)$ and $g_s(\rho)$ are obtained by means of the second iterative process [8]. For this purpose, we seek the differences $v - v_0$ and $u - u_0$ ($v_0 = 0$, $u_0 = 2\rho$) in the form

$$v = \sum_{s=0}^{n} \varepsilon^s h_s, \qquad u - 2\rho = \sum_{s=0}^{n} \varepsilon^s g_s \tag{2.2}$$

Substitute (2.2) into (1.1) and (1.2), and perform the change of variable $\rho = 1 - r$; then expand all coefficients in Taylor series about the point $r = 0$, and set $r = \varepsilon t$. Now setting the coefficients of $\varepsilon^0, \varepsilon^1, \ldots, \varepsilon^n$ equal to zero, we obtain a nonlinear system of Eqs. in h_0 and g_0

$$h_0'' + \tfrac{1}{2} g_0^2 + g_0 = 0, \qquad g_0'' - g_0 h_0 - h_0 = 0 \tag{2.3}$$

while for h_s, g_s ($s = 1, 2, \ldots, n$) we obtain

$$-h_s'' - g_s(1 + g_0) = \frac{1}{2} \sum_{\substack{k+m=s \\ (k,\, m\neq 0)}} g_k g_m - t h_{s-1}'' - h_{s-1}' - \sum_{k+m+2=s} t^k h_m + t g_{s-1} \tag{2.4}$$

$$-g_s'' + h_s(1 + g_0) + g_s h_0 = -\sum_{\substack{k+m=s \\ (k,\, m\neq 0)}} h_k g_m - t g_{s-1}'' - g_{s-1}' - \sum_{k+m+2=s} t^k g_m - t h_{s-1}$$

Similarly, from (1.2), we obtain the first boundary condition for h_0 and g_0 when $t = 0$; the second boundary condition is obtained from the requirement that the solution possess a boundary layer character in the neighborhood of $\rho = 1$, i.e.

$$g_0'(0) = 0, \quad h_0'(0) = 0, \quad g_0(\infty) = 0, \quad h_0(\infty) = 0 \tag{2.5}$$

From (2.3) and (2.5), it follows immediately that

$$h_0 = g_0 = 0 \tag{2.6}$$

Now, from (2.4), setting $s = 1$ and utilizing (2.6), we obtain

$$h_1'' + g_1 = 0, \qquad g_1'' - h_1 = 0$$

with the boundary conditions

$$g_1'(0) = 2(1+\mu), \quad h_1'(0) = 0; \qquad g_1(\infty) = h_1(\infty) = 0$$

whence

$$h_1'(\rho) = -\sqrt{2}(1+\mu)\exp\left(-\frac{1-\rho}{\sqrt{2}\,\varepsilon}\right)\left(\cos\frac{1-\rho}{\sqrt{2}\,\varepsilon} + \sin\frac{1-\rho}{\sqrt{2}\,\varepsilon}\right)$$
$$g_1(\rho) = -\sqrt{2}(1+\mu)\exp\left(-\frac{1-\rho}{\sqrt{2}\,\varepsilon}\right)\left(\cos\frac{1-\rho}{\sqrt{2}\,\varepsilon} - \sin\frac{1-\rho}{\sqrt{2}\,\varepsilon}\right) \tag{2.7}$$

The functions h_s and g_s ($s \geqslant 2$) are obtained from systems of linear differential Eqs. with constant coefficients

$$h_s'' + g_s = f_{1s}(t), \qquad g_s'' - h_s = f_{2s}(t) \tag{2.8}$$

where $f_{1s}(t)$ and $f_{2s}(t)$ are finite polynomials consisting of terms of the form

$$t^m\left(B\sin\frac{\sqrt{2}}{2}lt + C\cos\frac{\sqrt{2}}{2}nt\right)\exp\left(-\frac{\sqrt{2}}{2}kt\right)$$

with m, k, l and n as integers not exceeding s. Note the boundary conditions

$$-h_s' = \mu\sum_{k+m=s-1} t^k h_m, \qquad g_s' = \mu\sum_{k+m=s-1} t^k g_m \qquad (t=0)$$
$$h_s(\infty) = g_s(\infty) = 0 \qquad (s = 2, 3, \ldots, n) \qquad (t=\infty) \tag{2.9}$$

It is easily seen that h_s and g_s will be boundary layer type functions [8].

Finally, we introduce the infinitely differentiable, monotonous functions $\alpha_s(\rho)$ and $\beta_s(\rho)$, to correct for the incompatibility (of exponential order of smallness) of h_s and g_s, respectively, in satisfying the boundary conditions (1.2) for $\rho = 0$

$$\alpha_s(\rho) = \begin{cases} -h_s(0) & (0 \leqslant \rho \leqslant 0.1) \\ 0 & (0.2 \leqslant \rho \leqslant 1) \end{cases}$$
$$\beta_s(\rho) = \begin{cases} -g_s(0) & (0 \leqslant \rho \leqslant 0.1) \\ 0 & (0.2 \leqslant \rho \leqslant 1) \end{cases}$$

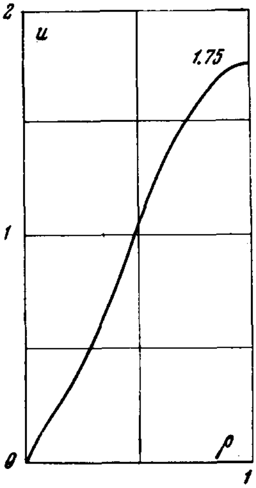

Fig. 1

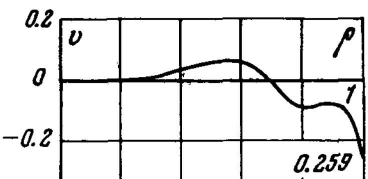

Fig. 2

Thus, the asymptotic expansions (2.1) may be written

$$v = \sum_{s=0}^{n}\varepsilon^s h_s + \sum_{s=0}^{n}\varepsilon^s\alpha_s + x_n,$$
$$u = 2\rho + \sum_{s=0}^{n}\varepsilon^s g_s + \sum_{s=0}^{n}\varepsilon^s\beta_s + z_n \tag{2.10}$$

Here, h_1 and g_1 are as defined in (2.7) while h_s and g_s ($s \geqslant 2$) are solutions of (2.8) and (2.9). Below, in Section 3, we will utilize the notation

$$\varphi_n = v - x_n, \qquad \psi_n = u - z_n \tag{2.11}$$

Note that, from (2.10) and the explicit expressions for h_s, g_s, α_s and β_s, we may readily obtain the estimates(*)

$$|\varphi_n| < m_1\varepsilon\rho, \qquad |\psi_n| < m_2\varepsilon\rho \tag{2.12}$$

As an example, we deduce φ_1 in terms of ρ. Thus,

$$\lim_{\rho\to 0}\frac{h_1(\rho)+\alpha_1(\rho)}{\rho} = \lim_{\rho\to 0}\frac{h_1(\rho)-h_1(0)}{\rho} = \left.\frac{dh_1}{d\rho}\right|_{\rho=0}$$

3. Substantiation of the asymptotic expansions. The existence of a nontrivial solution. We introduce the vector space $V \equiv (v, u)$, consisting of:

*) Here and hereafter m_i and c_i are certain positive constants independent of ε.

1) Vectors with the finite norm

$$(L_2) \qquad \|\mathbf{V}\|^2_{L_2} = \int_0^1 (v^2 + u^2)\, d\rho$$

2) The closure of the set of smooth vector-functions satisfying conditions (1.2), with the norm (H)

$$\|\mathbf{V}\|^2_H = \int_0^1 [(Av)^2 + (Au)^2]\, d\rho$$

Problem (1.1), (1.2) will be considered as the functional Eq.

$$P(V) = 0 \tag{3.1}$$

where the operator P is defined by the left-hand side of system (1.1).

The operator P maps from the H space into the L_2 space. In order to show this, we will need the following estimates:

$$\int_0^1 (v^4 + u^4)\, d\rho \leqslant m_3 \|\mathbf{V}\|^4_H, \qquad \max(|v| + |u|) \leqslant m_3 \|\mathbf{V}\|_H \quad (0 \leqslant \rho \leqslant 1) \tag{3.2}$$

We will now prove inequalities (3.2). Consider the differential Eqs.

$$\begin{aligned} Av &= f_1, \qquad \left[\frac{dv}{d\rho} - \frac{\mu}{\rho} v\right]_{\rho=1} = 0, \qquad \frac{v}{\rho}\Big|_{\rho=0} < \infty \\ Au &= f_2, \qquad \left[\frac{du}{d\rho} + \frac{\mu}{\rho} u\right]_{\rho=1} = 0, \qquad \frac{u}{\rho}\Big|_{\rho=0} < \infty \end{aligned} \tag{3.3}$$

It is easily seen that the Eqs. in (3.3) are, respectively, equivalent to the integral relations

$$\begin{aligned} v &= \frac{1}{\rho} \int_0^\rho \eta\, d\eta \int_\eta^1 \frac{f_1}{\xi}\, d\xi + \rho\, \frac{1+\mu}{1-\mu} \int_0^1 \eta\, d\eta \int_\eta^1 \frac{f_1}{\xi}\, d\xi \\ u &= \frac{1}{\rho} \int_0^\rho \eta\, d\eta \int_\eta^1 \frac{f_2}{\xi}\, d\xi + \rho\, \frac{1-\mu}{1+\mu} \int_0^1 \eta\, d\eta \int_\eta^1 \frac{f_2}{\xi}\, d\xi \end{aligned} \tag{3.4}$$

Now, (3.2) is obtained from (3.4) by the double application of the Buniakowski inequality.

Theorem 3.1. Problem (1.1), (1.2) has, in addition to the trivial solution $v = u \equiv 0$, a second solution for which the asymptotic expansions (2.10) are valid, whereupon the following estimates hold:

$$\max |x_n(\rho)| \leqslant m_4 \varepsilon^n$$

$$\max |z_n(\rho)| \leqslant m_4 \varepsilon^n \quad (n = 1, 2, \ldots) \quad (0 \leqslant \rho \leqslant 1) \tag{3.5}$$

To show existence, we make use of Kantorovich's theorem [9] concerning the convergence of Newton's method for operator equations, similarly to [2]. As a first approximation, we use the truncated asymptotic series $\mathbf{V}_k{}^* = (\varphi_k, \psi_k)$.

From Kantorovich's theorem, one easily obtains [2 and 3] the following theorem leading to the proof that there exists in the neighborhood of $\mathbf{V}_k{}^*$ a solution to (3.1) with the asymptotic representation $\mathbf{V}_k{}^*$.

Theorem 3.2. Suppose that the operator $\mathbf{P}$ is defined in the sphere $\Omega(\|\mathbf{V} - \mathbf{V}_k{}^*\| \leqslant R)$ of the H space, and has a continuous second derivative in the closed sphere $\Omega_0(\|\mathbf{V} - \mathbf{V}_k{}^*\| \leqslant r < R)$. Suppose further that there exists an operator $\Gamma_\varepsilon(\mathbf{V}) = [\mathbf{P}_{V_k^*}{}'(\mathbf{V})]^{-1}$, and the following conditions are satisfied:

$$1) \quad \|\mathbf{P}(\mathbf{V}_k{}^*)\|_{L_2} \leqslant C_1 \varepsilon^{k+1}, \qquad 2) \quad \|\mathbf{P}_{\mathbf{V}}{}''\| \leqslant C_3 \tag{3.6}$$

$$3) \quad \|\Gamma_\varepsilon\|_{(L_2 \to H)} \leqslant C_2 \varepsilon^{-m} \qquad (2m < k + 1) \tag{3.7}$$

Then $\mathbf{V}^*$ is a solution of (3.1) for sufficiently small ε

$$\varepsilon < (2C_1 C_2{}^2 C_3)^{2m-k-1}$$

and the following estimate holds:

$$\|\mathbf{V}^* - \mathbf{V}_k{}^*\|_H \leqslant C \varepsilon^{k+1-m}$$

We will show that the conditions of Theorem 3.2 are satisfied with $m = 4$, independently of k, and k may be chosen so that $k > 2m - 1$.

The first estimate (3.10) follows directly from the relations

$$\varepsilon^2 A\varphi_k - {}^1/_2\psi_k{}^2 + \rho\psi_k = O(\varepsilon^{k+1}), \qquad \varepsilon^2 A\psi_k + \varphi_k\psi_k - \rho\varphi_k = O(\varepsilon^{k+1})$$

which are easily established by substituting φ_k and ψ_k (see (2.11)) into the left-hand side of (1.1) and (1.2).

Further, we will show that the following estimate holds:

$$\| \Gamma_\varepsilon \|_{(L_2 \to H)} \leqslant C_2\varepsilon^{-4} \tag{3.8}$$

For this purpose, we consider the Fréchet derivative of the element $V_k{}^*$

$$P_{V_k^*}{}' (V) \equiv (\varepsilon^2 Av - \psi_k u + \rho u, \ \varepsilon^2 Au + \psi_k v + \varphi_k u - \rho v)$$

Consider the system of Eqs.

$$P_{V_k^*}{}' (V) = f, \qquad f \equiv (f_1, f_2) \tag{3.9}$$

with boundary conditions (1.2). With the aid of (2.10) and (2.11), (3.9) may be written in the form

$$\varepsilon^2 Av - \rho u + \varepsilon s_1 u = f_1, \quad \varepsilon^2 Au + \rho v - \varepsilon s_1 v + \varepsilon s_2 u = f_2 \tag{3.10}$$

Here

$$s_1 = \varepsilon^{-1}(\psi_k - 2\rho), \qquad s_2 = \varepsilon^{-1}\varphi_k.$$

Multiplying the first Eq. in (3.10) by $(v - u)$ and the second by $(v + u)$, then integrating over the range zero to unity and combining, we obtain

$$\varepsilon^2 \int_0^1 \left(\rho v'^2 + \frac{v^2}{\rho} + \rho u'^2 + \frac{u^2}{\rho}\right) d\rho + \varepsilon^2\mu u^2(1) - \varepsilon^2\mu v^2(1) + \int_0^1 \rho(u^2 + v^2)\, d\rho -$$

$$- \varepsilon \int_0^1 [s_1(u^2 + v^2) - s_2 u(v + u)]\, d\rho = \int_0^1 [f_1(v - u) + f_2(v + u)]\, d\rho - 2\varepsilon^2\mu v(1)\, u(1) \tag{3.11}$$

Note that in obtaining (3.11) we must make use of an equation which holds for all smooth functions satisfying the boundary conditions (1.2):

$$\int_0^1 Au\, v\, d\rho - \int_0^1 Av\, u\, d\rho = 2\mu v(1)\, u(1) \tag{3.12}$$

The above Eq. can be proved by integration by parts. Utilizing (1.2), we obtain

$$\int_0^1 Av\, u\, d\rho = -u(1)\, v(1) - u(1)\, v'(1) + \int_0^1 \frac{1}{\rho} \frac{d}{d\rho}(\rho v) \frac{d}{d\rho}(\rho u)\, d\rho$$

Interchanging u and v and subtracting one equation from the other, we find that

$$\int_0^1 Au\, v\, d\rho - \int_0^1 Av\, u\, d\rho = v'(1)\, u(1) - u'(1)\, v(1)$$

Utilizing (1.2), we arrive at (3.12). The right-hand side of (3.11) may be estimated from Expression

$$\int_0^1 (|f_1| + |f_2|)(|v| + |u|)\, d\rho + \varepsilon^2\mu(v^2(1) + u^2(1)) \tag{3.13}$$

Furthermore, it follows from (2.12) that, for sufficiently small ε, the following estimates hold:

$$|s_1(\rho, \varepsilon)| < 4(1 + \mu)\rho, \qquad |s_2(\rho, \varepsilon)| < 4(1 + \mu)\rho \tag{3.14}$$

Applying (3.14) together with the obvious inequalities $2uv \leqslant u^2 + v^2$ and $u^2 \leqslant u^2 + v^2$ to the third integral in the left-hand side of (3.11), we obtain

$$J \equiv -\varepsilon \int_0^1 [s_1(v^2 + u^2) - s_2 u(v + u)]\, d\rho \leqslant 10\varepsilon(1 + \mu) \int_0^1 \rho(v^2 + u^2)\, d\rho \tag{3.15}$$

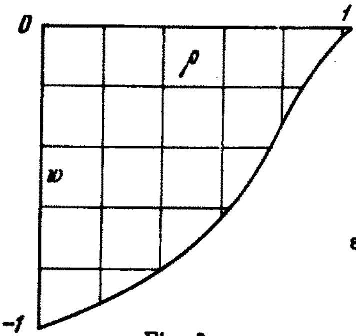

Fig. 3

We now find that

$$\int_0^1 \rho\,(v^2+u^2)\,d\rho + J \geqslant \frac{1}{2}\int_0^1 \rho\,(v^2+u^2)\,d\rho \qquad (3.16)$$

$$\text{for}\quad \varepsilon < \frac{1}{20\,(1+\mu)}$$

Taking note of (3.16) and (3.13), we obtain from (3.11)

$$\varepsilon^2\int_0^1\left(\rho v'^2+\frac{v^2}{\rho}+\rho u'^2+\frac{u^2}{\rho}\right)d\rho+\frac{1}{2}\int_0^1\rho\,(v^2+u^2)\,d\rho\leqslant$$

$$\leqslant\int_0^1(|f_1|+|f_2|)(|v|+|u|)\,d\rho+2\varepsilon^2\mu v^2(1) \qquad (3.17)$$

Making use of the obvious inequality

$$\int_0^1\left(\rho v'^2+\frac{v^2}{\rho}\right)d\rho\geqslant 2\int_0^\rho vv'\,d\rho=v^2(1)$$

we obtain from (3.17)

$$\varepsilon^2(1-2\mu)\int_0^1\left(\rho v'^2+\frac{v^2}{\rho}+\rho u'^2+\frac{u^2}{\rho}\right)d\rho+\frac{1}{2}\int_0^1\rho\,(v^2+u^2)\,d\rho\leqslant$$

$$\leqslant\int_0^1(|f_1|+|f_2|)(|v|+|u|)\,d\rho \qquad \left(0<\mu<\frac{1}{2}\right)$$

Whence, we have over the interval $0\leqslant\rho\leqslant 1$

$$\varepsilon^2(1-2\mu)(\max|u|^2+\max|v|^2)\leqslant 2\,\|\mathbf{f}\|_{L_2}\|\mathbf{V}\|_{L_2}\leqslant 2\,\|\mathbf{f}\|_{L_2}(\max|u|^2+\max|v|^2)^{1/2} \qquad (3.18)$$

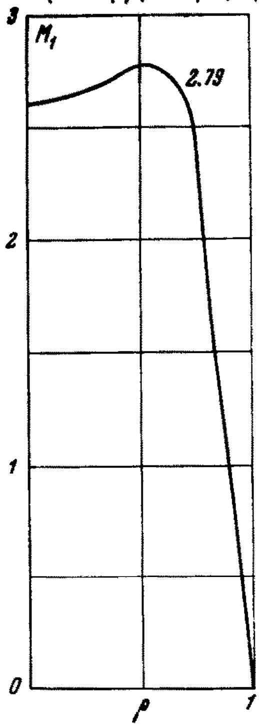

Fig. 4

From (3.18), we obtain

$$\max_{0\leqslant\rho\leqslant1}|u|+\max_{0\leqslant\rho\leqslant1}|v|\leqslant 4\varepsilon^{-2}(1+2\mu)^{-1}\|\mathbf{f}\|_{L_2} \qquad (0\leqslant\rho\leqslant1)$$

Now we obtain from (3.10) an estimate in H. We have

$$Av=\varepsilon^{-2}(f_1+\rho u-\varepsilon s_1 u),$$

$$Au=\varepsilon^{-2}(f_2-\rho v+\varepsilon s_1 v-\varepsilon s_2 u)$$

Utilization of (3.18) leads to

$$|Av|\leqslant\varepsilon^{-2}\left[|f_1|+\frac{5\,\|\mathbf{f}\|_{L_2}}{\varepsilon^2(1-2\mu)}\right],$$

$$|Au|\leqslant\varepsilon^{-2}\left[|f_2|+\frac{5\,\|\mathbf{f}\|_{L_2}}{\varepsilon^2(1-2\mu)}\right] \qquad (3.19)$$

Finally, application of (3.19) yields

$$\|\mathbf{V}\|_H\leqslant C_2\varepsilon^{-4}\|\mathbf{f}\|_{L_2},\qquad \|\mathbf{V}\|_H\leqslant C_2\varepsilon^{-4}\|\mathbf{P}_{\mathbf{V}_k^*}'(\mathbf{V})\|_{L_2}$$

Whence, it is readily found that the operator $\mathbf{P}_{\mathbf{V}_k^*}'$ has an inverse, and the estimate (3.8) holds.

Consider the bilinear form

$$\mathbf{P}''(\mathbf{V}')(\mathbf{V}'')\equiv(-u'u'',\ u'v''+v'u'')$$

Applying (3.2), we obtain $\|\mathbf{P}''(\mathbf{V}')(\mathbf{V}'')\|_{L_2}\leqslant C_3\|\mathbf{V}'\|_H\|\mathbf{V}''\|_H$. Whence, the second estimate in (3.6) follows.

Thus, the conditions of Theorem 3.2 are satisfied if $k>7$ and ε is sufficiently small ($0<\varepsilon<\varepsilon_1$). Therefore, (3.1) has a solution $\mathbf{V}^*\equiv(v,u)$, for which the following estimate holds:

$$\|\mathbf{V}^*-\mathbf{V}_k^*\|_H\leqslant m\varepsilon^{k-3}\qquad(k>7) \qquad (3.20)$$

Now, employing the triangle inequality, the theorem for embedding C in H (see (3.2)) and the explicit expressions for h_s and g_s, we obtain (3.5) from (3.20). Note also that the estimates for max $|x_n'(\rho)|$ and max$|z_n'(\rho)|$ can be obtained in a similar manner.

4. Example. The asymptotic expansions (2.10) provide very simple formulas for the evaluation of the fundamental characteristic value for the second equilibrium form. Let $H/h = 8$, where H is the shell height. Then $\varepsilon^2 = \frac{1}{2} h/H\gamma = 0.141$ ($\mu = 0.3$, $a^2 = 2RH$).

The quantities v and u are calculated within accuracy of order ε, inclusively, by means of Formulas (2.10) and (2.7) (Figs. 1 and 2).

The deflection and moment are obtained from the Formulas (Figs. 3 and 4)

$$w(\rho) = \int_1^\rho u d\rho, \quad M_1 = \frac{du}{d\rho} + \frac{\mu}{\rho} u$$

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